RAMAKRISHNA MISSION VIDYAMANDIRA

(Residential Autonomous College under University of Calcutta)

P.G. FIRST YEAR

M.A./M.SC. FIRST SEMESTER (July – December), 2012

Mid-Semester Examination, October, 2012

Date : 08/10/2012 **MATHEMATICS**

Time : 11 am – 3 pm Full Marks : 100

[Use separate answer scripts for each group]

Group – A (S.K.C)

(*Answer **any five** questions*) [5×5]

1. Show that $\mathbb{Z}_{pq}$, where p and q are two distinct primes, is decomposable as direct sum of two of its cyclic subgroups of orders p and q; but the dihedral group D_4 is indecomposable.
2. Let a group G acts on a set A and $a \in A$. Define orbit and stabilizer of a. Establish that order of the orbit of a is equal to the index of stabilizer of a in G.
3. If the order of a group G does not divide n! where n is the index of a subgroup H of G, what can you say about the nature of the subgroup H? Using generalised Cayley's theorem, justify your answer. Hence show that any group of order pn, p a prime and p > n, having a subgroup of order p is not simple.
4. State Cauchy's theorem related to a finite group. Use the theorem to prove that the converse of Lagrange's theorem for finite group holds in case of a finite abelian group.
5. Justify completely the statement : "Any group of order 99 is solvable".
6. Explain the concepts of upper central series of a group and a nilpotent group. Justify that the dihedral group D_4 is nilpotent and solvable.
7. a) Find a composition series of $S_3 \times \mathbb{Z}_2$. Is the group solvable?

   b) Find two composition series in the group $(\mathbb{Z}_{24}, +_{24})$ and verify Jordan-Holder theorem.

Group – B (S.K.G)

(*Answer **Q.No. 8** and **any two** from the rest*)

8. Let τ be the cocountable topology on $\mathbb{R}$. If A = Q, B = R – Q, $C = (0,1) \cup \{2,3,4,5,....\}$. Find the following (no justification is needed)

   a) $\bar{A}, A^o, A^d$

   b) $\bar{B}, B^o, B^d$

   c) $\bar{C}, C^o, C^d$ [3+3+3]

9. a) In R with usual topology, find two countable, disjoint, dense sets. Justify your answer.

   b) Prove that an uncountable space with cofinite topology is separable and Lindelöf but not 1st countable. [3+5]

10. a) Define 'open hereditary property' of a topological space.

    b) State and prove pasting lemma.

    c) Use pasting lemma to prove that the function $f: R \to R$ defined by $f(x) = |x|\ \forall x \in R$ is continuous. [1+4+3]

11. Let τ be the cofinite topology and τ_u be the usual topology on R. Let $f:(R,\tau) \to (R,\tau_u)$ be defined by $f(x) = x^2\ \forall x \in R$. Verify with proper reason whether the following are true.

    a) f is an open map.

b) f is a closed map.

c) f is a continuous map.

d) There is a homeomorphism from (R,τ) to (R,T_u). [4×2]

Group – C (P.S.M)

(*Answer **any five** questions*) [5×5]

12. Let F_3 be a finite field of 3 elements. How many 2×2 invertible matrices with entries from F_3 are there? Let us consider the set of all $T:\mathbb{R}^2\to\mathbb{R}^2$ such that T preserves the line y = x. What is the dimension of this subspace of $M_{2\times2}(\mathbb{R})$. (Give a basis). [3+2]

13. Give an example of a linear transformation $T:\mathbb{R}^2\to\mathbb{R}^2$ such that N(T) = R(T), where N(T) is the nullity and R(T) is the rank of T. Find the matrix of reflection (in $\mathbb{R}^2$) about the θ-line and prove that two reflections bring back the original. [2+3]

14. Consider the linear transformations diff : $P_3(R)\to P_2(R)$ and Int : $P_2(R)\to P_3(R)$. Using the standard basis justify that differentiation is a left-inverse of integration but not conversely. Justify that every matrix transforms its row space onto its column space. [3+2]

15. Let $A=\begin{bmatrix}1 & 2 & 0 & 2 & 5\\ -2 & -5 & 1 & -1 & -8\\ 0 & -3 & 3 & 4 & 1\\ 3 & 6 & 0 & -7 & 2\end{bmatrix}$ and $U=\begin{bmatrix}1 & 0 & 2 & 0 & 1\\ 0 & 1 & -1 & 0 & 1\\ 0 & 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$ be the reduced row-echelon matrix from A. Find a basis of row space, column space and null space of A. Use Gauss-Jordan elimination to evaluate the determinant. $P=\begin{vmatrix}1 & 7 & -9\\ 3 & -5 & 2\\ 2 & 14 & -18\end{vmatrix}$. [3+2]

16. Let V be a finite dimensional vector space over $\mathbb{R}$. Prove that each basis of its dual space V* is the dual of some basis of V. [5]

17. Let $V=P_2(\mathbb{R})$ and for three real numbers $t_1\neq t_2\neq t_3$, let $L_i(p)=p(t_i)\,\forall i=1,2,3$ be three linear functionals on v. Prove that $\{L_1, L_2, L_3\}$ is a basis of V*. What is the basis of V for which this is the dual basis? Let W be a finite dimensional vector space and V be a vector subspace of W. Then prove that $\dim W/V=\dim W-\dim V$. [1+2+2]

18. Let$\{c_1, c_2,\ldots c_n)\}$ be n column vectors of an m×n matrix A.

    Define $\mathcal{P}(A)=\left\{\sum_{i=1}^{n}t_ic_i : 0\le t_i\le 1, i=1,2,3...,n\right\}$ and prove that, $\mathrm{vol}(\mathcal{P}(A))=\sqrt{\det A^TA}$.

    Let W be a subspace of a vector space V, and let Q be the quotient mapping of V onto V/W. Suppose W' is a subspace of V. Then show that $V=W\oplus W'$ iff the restriction of Q to W' is an isomorphism of W' onto V/W. [3+2]

Group – D (Bhargav Mj.)

(*Answer **any five** questions*) [5×5]

19. Let $f:\mathbb{R}\to\mathbb{R}$ be a map that sends Cauchy sequences into Cauchy sequences. Is f necessarily (i) continuous? (ii) uniformly continuous? Justify your answer.

20. Find lim sup and lim inf of the sequence $\{s_n\}$ defined by $s_1 = 0$; $s_{2m}=\frac{s_{2m-1}}{2}$; $s_{2m+1}=\frac{1}{2}+s_{2m}$ for $m\in\mathbb{N}$.

21. Let $\{K_\alpha\}$ be a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty. Prove that $\bigcap K_\alpha\neq\phi$.

22. Let $a_n > 0$ and $s_n = \sum_{i=1}^{n} a_i$ for $n \geq 1$. Assuming that $\sum a_n$ is divergent, check for convergence of $\sum \frac{a_n}{1+na_n}$ and $\sum \frac{a_n}{1+n^2 a_n}$.

23. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous at every irrational point and is discontinuous at every rational point (in $\mathbb{R}$).

24. State and prove Banach fixed point theorem.

25. Show that the uniform limit of a sequence of continuous functions is continuous.

26. Characterize all connected sets in $\mathbb{R}$ (with usual metric).